

# Bound states for one-electron atoms in higher dimensions

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## Abstract

We study the Schrödinger equation for one-electron atoms in space-times with  $d \geq 4$  spatial dimensions where the Gauss law is assumed to be valid. It is shown that there are no normalizable wave functions corresponding to bound states. The consistency with the classical limit is discussed.

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## I. INTRODUCTION

Currently, string theory seems to be the best candidate for a quantum theory of gravity or possibly for a unified theory of fundamental interactions. Consistency of string theory, in the standard formulations, requires the existence of extra spatial dimensions [1, 2]. Presently, there is no experimental evidence of such extra dimensions. This indicates that, if they exist, they are compactified into small length scales still inaccessible to our measurements.

One way to find upper bounds for the “sizes” of these extra dimensions is to consider microscopic physical systems for which we have a very well established agreement between theoretical modeling and experimental data. Then, one looks at the corresponding version of the physical system assuming the existence of non-compact extra dimensions at that length scale. One interesting example is the hydrogen atom (or, more generally, a one-electron atom).

Physical systems subject to Gaussian central forces in different spatial dimensions were discussed a long time ago by Ehrenfest [3]. He showed that a classical planetary system does not have stable orbits for spaces with more than three non-compact spatial directions. He also discussed the Bohr atomic model in such spaces.

The quantum mechanical spectrum of a one electron atom in higher dimensional spaces, assuming the Gauss law to hold, was discussed in ref. [4]. There, it was shown that there is no lower bound for negative energy solutions. As a consequence, there would be no stable atom in such spaces.

Many recent articles (see, e.g., [5, 6]) disregard Gauss law when discussing the hydrogen atom or its supersymmetric extension in higher dimensional spaces, taking the electrostatic potential to fall as  $1/r$ . In string theory, such approach is consistent with the idea of some brane-world scenarios, where Standard Model fields, such as the electromagnetic, live on a 3+1 dimensional D3-brane while gravity propagates in the extra dimensions.

Here we consider a model where space-time has one timelike dimension and  $d$  spatial dimensions and assume that Gauss law holds for the electric field, i.e., the total electric flux on a closed surface in the  $d$  dimensional space is proportional to the charge. This approach is consistent with a scenario where the electromagnetic field is not constrained to a D3-brane. We will study the Schrödinger equation for one electron atoms and search for normalizable wave functions corresponding to bound states, i.e., negative energy eigenvalues. We will see

that the  $d = 4$  case must be handled in a particular way.

## II. SCHRÖDINGER EQUATION

The electric field produced by a static nucleus with  $Z$  protons of elementary charge  $q$  in a space with  $d$  non-compact spatial dimensions depends on the distance  $r$  as [2]:

$$\vec{E}(\vec{r}) = \frac{\Gamma(d/2)}{2\pi^{d/2}} \frac{Zq}{r^{d-1}} \hat{r} . \quad (1)$$

The corresponding static potential, taking as usual the potential at infinity to vanish, reads

$$V(r) = \frac{\Gamma(d/2)}{2\pi^{d/2}(d-2)} \frac{Zq}{r^{d-2}} \equiv \eta_d \frac{q}{r^{d-2}} . \quad (2)$$

Let us assume the existence of a stationary state (energy eigenstate) for an electron of mass  $\mu$  with negative energy  $\mathcal{E}$ , and study the solutions of the corresponding Schrödinger equation

$$\left[ \frac{p^2}{2\mu} + U(r) \right] \Phi_{\mathcal{E}}(\vec{r}) = \mathcal{E} \Phi_{\mathcal{E}}(\vec{r}) \quad (3)$$

where  $U(r) = -q V(r)$

We are dealing with a central potential, so it is convenient to write the eigenfunction as usual:

$$\Phi_{\mathcal{E}}(\vec{r}) = R_{\mathcal{E}}(r) Y_{\mathcal{E}}(\Omega) . \quad (4)$$

The operator  $p^2$  can be shown to act in this space with  $d$  spatial dimensions as [7]

$$p^2 = p_r^2 + \frac{1}{r^2} \left( L^2 + \frac{\hbar^2}{4} (d-1)(d-3) \right) , \quad (5)$$

where the radial component  $p_r$  acts on the radial part of the wave function as

$$p_r^2 R(r) = -\hbar^2 r^{-\frac{d-1}{2}} \frac{d^2}{dr^2} \left( r^{\frac{d-1}{2}} R(r) \right) , \quad (6)$$

while the angular momentum operator acts on the angular part as [7]

$$L^2 Y_{\ell,m}(\Omega) = \hbar^2 \ell(\ell+1+d-3) Y_{\ell,m}(\Omega) . \quad (7)$$

Equation (6) suggests that we write the radial eigenfunctions as

$$R_{\mathcal{E},\ell}(\vec{r}) = r^{-\frac{d-1}{2}} f_{\mathcal{E},\ell}(r) \quad (8)$$

so that the radial equation takes the form:

$$\left[ -\frac{d^2}{dr^2} + \frac{\ell_d(\ell_d + 1)}{r^2} + \frac{2m}{\hbar^2}(U(r) - \mathcal{E}) \right] f_{\mathcal{E},\ell}(r) = 0, \quad (9)$$

where we introduced, for simplicity,

$$\ell_d = \ell + (d - 3)/2. \quad (10)$$

This equation may be written in terms of dimensionless quantities

$$\left[ \frac{d^2}{d\rho^2} - \frac{\ell_d(\ell_d + 1)}{\rho^2} + \frac{\eta_d}{\rho^{d-2}} - \left( \alpha^{\frac{3-d}{4-d}} \lambda_{\mathcal{E}} \right)^2 \right] f_{\mathcal{E},\ell}(\rho) = 0 \quad (11)$$

where

$$\alpha = \frac{mq^2}{\hbar^2} ; \quad \rho = \alpha^{\frac{1}{4-d}} r ; \quad \lambda_{\mathcal{E}}^2 = -\frac{2\hbar^2}{mq^4} \mathcal{E}.$$

At this point we note that this adimensionalization procedure, well known for the three dimensional case, does not work for  $d = 4$ .

### III. CASE $d > 4$

First, let us handle the case  $d \neq 4$ . Making the new substitution

$$f_{\mathcal{E},\ell}(\rho) = \exp\{-\alpha^{\frac{3-d}{4-d}} \lambda_{\mathcal{E}} \rho\} y_{\mathcal{E},\ell}(\rho) = 0 \quad (12)$$

we obtain

$$\left[ \frac{d^2}{d\rho^2} - 2\alpha^{\frac{3-d}{4-d}} \lambda_{\mathcal{E}} \frac{d}{d\rho} + \left( \frac{\eta_d}{\rho^{d-2}} - \frac{\ell_d(\ell_d + 1)}{\rho^2} \right) \right] y_{\mathcal{E},\ell}(\rho) = 0. \quad (13)$$

Considering the fact that the wave function must be analytic, we can search for a power series solution:

$$y_{\mathcal{E},\ell}(\rho) = \sum_{n=0}^{\infty} c_n \rho^{n+s}.$$

Equation (13) gives us

$$\begin{aligned} c_0 &= \dots = c_{d-4} = 0 \\ c_{n+d-3} &= \frac{\ell_d(\ell_d + 1) - (n+s)(n+s-1)}{\eta_d} c_{n+1} + \frac{2\alpha^{\frac{3-d}{4-d}} \lambda_{\mathcal{E}}(n+s)}{\eta_d} c_n \end{aligned} \quad (14)$$

Hence, all the coefficients  $c_n$  vanish and there is no non-trivial solution to  $y_{\mathcal{E},\ell}$  in this case.

#### IV. CASE $d = 4$

The case  $d = 4$  must be dealt with differently. From (2) we have

$$U(r) = -\frac{Zq^2}{4\pi^2 r^2}. \quad (15)$$

The radial time-independent Schrödinger equation becomes

$$\left[ \frac{d^2}{dr^2} + \frac{3}{r} \frac{d}{dr} - \left( \ell_4(\ell_4 + 1) - \frac{3}{4} - \frac{mZq^2}{2\pi^2 \hbar^2} \right) \frac{1}{r} + \frac{2m\mathcal{E}}{\hbar^2} \right] R(r) = 0. \quad (16)$$

Since we are only interested in bound states, it is convenient to introduce:

$$\epsilon^2 = -\frac{2m\mathcal{E}}{\hbar^2}. \quad (17)$$

Now, making the substitutions

$$\rho = \epsilon r, \quad R(\rho) \equiv \frac{\epsilon}{\rho} g(\rho), \quad \zeta_\ell = 1 + \ell_4(\ell_4 + 1) - \frac{3}{4} - \frac{mZq^2}{2\pi^2 \hbar^2},$$

the radial equation takes the form

$$\left[ \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + -\left( 1 + \frac{\zeta_\ell}{\rho^2} \right) \right] g(\rho) = 0. \quad (18)$$

The sign of the parameter  $\zeta_\ell$  depends on  $\ell_4 = \ell + 1/2$ .

Considering first the case of negative sign, we write  $\zeta = -\nu^2$ , with  $\nu$  real and obtain

$$\left[ \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + -\left( 1 - \frac{\nu^2}{\rho^2} \right) \right] g(\rho) = 0. \quad (19)$$

Searching again for a series solution  $\sum_{n=0}^{\infty} c_n \rho^{n+s}$  for this equation we find the conditions

$$\begin{aligned} (s^2 + \nu^2)c_0 &= 0 \\ ((s+1)^2 + \nu^2)c_1 &= 0 \\ c_n &= \frac{1}{\nu^2 + (n+s)^2} c_{n-2} \end{aligned} \quad (20)$$

and, since  $\nu^2$  is always non-zero, we conclude that there is no non-trivial solution.

Now, considering  $\zeta_\ell \geq 0$ , the corresponding equation (using  $\zeta = \nu^2$ ) is

$$\left[ \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + -\left( 1 + \frac{\nu^2}{\rho^2} \right) \right] g(\rho) = 0. \quad (21)$$

This has the solutions  $I_\nu(\rho)$  and  $K_\nu(\rho)$  (modified Bessel functions). But the first diverges at infinity, while the latter diverges at the origin. So, there is no physically acceptable (normalizable) bound state solution for the Schrödinger equation in this  $d = 4$  case.

## V. CONCLUSION

We conclude that, assuming Gauss law still holds, there cannot be any bound (negative energy) solution for the Schrödinger equation of the one-electron atom in higher non-compact dimensions. We point out that this result is consistent with that of its classical gravitational analogue, the Kepler problem. In such systems, for a given  $d \geq 4$ , there is only one stationary state, which disrupts under the smallest perturbation. That is, the classical stationary states, corresponding to unstable ( $d > 4$ ) or neutral ( $d = 4$ ) equilibrium, must satisfy the conditions:

$$\Delta r = 0 \quad ; \quad \Delta p_r = 0 \quad .$$

In quantum mechanics the uncertainty relations forbid such a situation, and the stationary states do not even occur.

Thus, the presence of atoms everywhere in our Universe places a restriction on the size of hypothetical extra dimensions. They should have to be smaller than the atomic scale unless the electromagnetic field does not propagate in these directions.

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